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# Bounds for determinants of meet matrices associated with incidence functions

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**Abstract**

We consider meet matrices on meet-semilattices as an abstract generalization of greatest common divisor (gcd) matrices. Some new bounds for the determinant of meet matrices and a formula for the inverse of meet matrices are given. © 2001 Elsevier Science Inc. All rights reserved.

*Keywords:* Meet matrix; Incidence function; Bounds for determinant; Inverse matrix; The Gram–Schmidt orthogonalization process; Smith's determinant

## 1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of *n* distinct positive integers. The $n \times n$ matrix $(S)$ having the greatest common divisor $(x_i, x_j)$ of $x_i$ and $x_j$ as its $i, j$-entry is called the greatest common divisor (gcd) matrix of *S*. The set *S* is said to be factor-closed if it contains every divisor of *x* for any $x \in S$. The set *S* is said to be gcd-closed if $(x_i, x_j) \in S$ for all $1 \leqslant i, j \leqslant n$. Clearly, a factor-closed set is gcd-closed but not conversely.

Let *f* be an arithmetical function and let $(f(x_i, x_j))$ denote the $n \times n$ matrix having *f* evaluated at the greatest common divisor $(x_i, x_j)$ of $x_i$ and $x_j$ as its $i, j$-entry. Let $C_S$ denote the class of arithmetical functions defined as

$$C_S = \{f \mid (x \in S, d \mid x) \;\Rightarrow\; (f * \mu)(d) > 0\},$$

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where $*$ is the Dirichlet convolution and $\mu$ is the number-theoretic Möbius function. Hong [2] showed that if $f \in C_S$, then

$$\det(f(x_i, x_j)) \geqslant \prod_{k=1}^{n} \sum_{\substack{d|x_k \\ d\nmid x_t \\ t<k}} (f * \mu)(d) \tag{1.1}$$

and the equality holds if and only if $S$ is gcd-closed. Hong [2, Theorem 3] also obtained an upper bound for $\det(f(x_i, x_j))$. (Note that this formula contains some errors.) In this paper we give abstract generalizations of these formulae considering bounds for determinants of meet matrices on meet-semilattices. Haukkanen [1] has previously studied meet matrices on meet-semilattices and this paper continues his work. Note that some notations differ from those used in [1].

## 2. Definitions

Let $(P, \leqslant)$ be a meet-semilattice such that the principal order ideal $\downarrow x = \{y \in P \mid y \leqslant x\}$ is finite for all $x \in P$.

Let $S$ be a subset of $P$. We say that $S$ is lower-closed if for every $x, y \in P$ with $x \in S$ and $y \leqslant x$, we have $y \in S$. We say that $S$ is meet-closed if for every $x, y \in S$, we have $x \wedge y \in S$. Obviously the concepts "lower-closed" and "meet-closed" are generalizations of the concepts "factor-closed" and "gcd-closed", respectively. It is also clear that a lower-closed set is always meet-closed but not conversely. The order ideal generated by $S$ is given as $\downarrow S = \{y \in P \mid \exists x \in S : y \leqslant x\}$. Obviously $\downarrow S$ is the minimal lower-closed set containing $S$.

Let $f$ be a complex-valued function on $P \times P$ such that $f(x, y) = 0$ whenever $x \nleqslant y$. Then we say that $f$ is an incidence function of $P$. If $f$ and $g$ are incidence functions of $P$, their sum $f + g$ is defined by $(f + g)(x, y) = f(x, y) + g(x, y)$ and their convolution $f * g$ is defined by $(f * g)(x, y) = \sum_{x \leqslant z \leqslant y} f(x, z)g(z, y)$. The set of all incidence functions of $P$ with addition and convolution forms a ring, where the identity $\delta$ is defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. The incidence function $\zeta$ is defined by $\zeta(x, y) = 1$ if $x \leqslant y$, and $\zeta(x, y) = 0$ otherwise. The Möbius function $\mu$ of $P$ is the inverse of $\zeta$.

In what follows, let $P$ be a meet-semilattice such that all principal order ideals of $P$ are finite. Furthermore, let $S$ be a finite subset of $P$, and denote $S = \{x_1, x_2, \ldots, x_n\}$ with $x_i < x_j \Rightarrow i < j$. For any incidence function $f$ of $P$ we denote $f(0, x) = f(x)$, where $0 = \min P$. For example, if $(P, \leqslant) = (\mathbf{Z}_+, |)$, then $\mu(1, n)$ is the usual number-theoretic Möbius function $\mu(n)$.

**Definition 2.1.** Let $C_S$ denote the class of incidence functions defined as

$$C_S = \{f \mid (x \in S, z \leqslant x) \Rightarrow (f * \mu)(z) > 0\}. \tag{2.1}$$

**Definition 2.2.** If $f$ is an incidence function of $P$, then the $n \times n$ matrix $(S)_f = (s_{ij})$, where

$$s_{ij} = f(x_i \wedge x_j), \tag{2.2}$$

is called the meet matrix on $S$ with respect to $f$.

## 3. Structure theorem

**Lemma 3.1.** *Let $f$ be an incidence function of $P$. Then*

$$f(x, y) = \sum_{x \leqslant z \leqslant y} (f * \mu)(x, z) \tag{3.1}$$

*for all $x, y \in P$.*

Lemma 3.1 is a direct consequence of the formula $f = f * \delta = f * (\mu * \zeta) = (f * \mu) * \zeta$.

**Lemma 3.2.** *Let $\downarrow S = \{y_1, y_2, \ldots, y_m\}$ with $y_i < y_j \Rightarrow i < j$, and let $f$ be an incidence function of $P$. Let $A$ denote the $n \times m$ matrix defined by*

$$a_{ij} = \begin{cases} \sqrt{(f * \mu)(y_j)} & \text{if } y_j \leqslant x_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

*Then $(S)_f = AA^{\mathrm{T}}$.*

**Proof.** Obviously $y_1 = \min P$. For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ we have

$$\begin{aligned} (AA^{\mathrm{T}})_{ij} &= \sum_{k=1}^{m} a_{ik} a_{jk} = \sum_{\substack{y_1 \leqslant y_k \leqslant x_i \\ y_1 \leqslant y_k \leqslant x_j}} (f * \mu)(y_1, y_k) \\ &= \sum_{y_1 \leqslant y_k \leqslant x_i \wedge x_j} (f * \mu)(y_1, y_k). \end{aligned}$$

Therefore it follows from Lemma 3.1 that

$$(AA^{\mathrm{T}})_{ij} = f(y_1, x_i \wedge x_j) = f(x_i \wedge x_j).$$

This completes the proof. $\square$

## 4. Determinant of meet matrices

Haukkanen [1] has proved that if $S$ is meet-closed, then

$$\det(S)_f = \prod_{k=1}^{n} \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f * \mu)(z). \tag{4.1}$$

Note that Haukkanen writes this formula without using convolution of incidence function. Also note that (4.1) is a generalization of Smith's [4] famous formula: if $S = \{x_1, x_2, \ldots, x_n\}$ is a factor-closed set of positive integers and $f$ is an arithmetical function, then $\det(f(x_i, x_j)) = \prod_{i=1}^{n} (f * \mu)(x_i)$.

## 5. Lower bound for $\det(S)_f$

In this section we give a generalization of (1.1). The proof is adapted from that given by Hong [2].

**Theorem 5.1.** *If $f \in C_S$, then*

$$\det(S)_f \geqslant \prod_{k=1}^{n} \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f * \mu)(z) \tag{5.1}$$

*and the equality holds if and only if S is meet-closed.*

**Proof.** Define $S_k = \{z \in P \mid z \leqslant x_k,\ z \nleqslant x_t,\ t < k\}$, $1 \leqslant k \leqslant n$. Then for all $1 \leqslant i < j \leqslant n$ we have $S_i \cap S_j = \emptyset$. Otherwise, there exists $z \in S_s \cap S_t$, where $s < t$. Since $z \in S_t$ and $s < t$, we have $z \nleqslant x_s$, and this contradicts $z \in S_s \cap S_t$. Obviously $S_1 \cup \cdots \cup S_n = \downarrow S$. To see this take $z \in \downarrow S$. Then for some $i$ we have $z \leqslant x_i$. From our assumptions we see that the interval $[z, x_i]$ is finite. We can therefore find the minimal $k$ such that $z \leqslant x_k$. Thus $z \nleqslant x_t$ when $t < k$. This means that $z \in S_k$ and we have $z \in S_1 \cup \cdots \cup S_n$.

For all $1 \leqslant k \leqslant n$ let $S_k = \{y_{k,1}, y_{k,2}, \ldots, y_{k,p_k}\}$ with $y_{k,i} < y_{k,j} \Rightarrow i < j$. Obviously $y_{k,p_k} = x_k$, $1 \leqslant k \leqslant n$. Let $p_1 + \cdots + p_n = m$, and let

$$y_j = \begin{cases} y_{1,j} & \text{if } j = 1, 2, \ldots, p_1, \\ y_{k,t} & \text{if } j = p_1 + \cdots + p_{k-1} + t, \quad 1 \leqslant t \leqslant p_k,\ k \geqslant 2, \end{cases}$$

for all $1 \leqslant j \leqslant m$. Then we have $\downarrow S = \{y_1, y_2, \ldots, y_m\}$ with $y_i < y_j \Rightarrow i < j$. To see the latter statement, let $y_i < y_j$. Then for $y_i$ there exist $d$ and $s$ such that $y_i = y_{d,s}$, where $i = p_1 + \cdots + p_{d-1} + s$, $1 \leqslant s \leqslant p_d$. In the same way, for $y_j$ there exist $e$ and $t$ such that $y_j = y_{e,t}$, where $j = p_1 + \cdots + p_{e-1} + t$, $1 \leqslant t \leqslant p_e$. Since

$y_{d,s} \leqslant y_{e,t}$, we have $d \leqslant e$. If $d = e$, we have trivially $i < j$. If $d < e$, then $i \leqslant p_1 + \cdots + p_d \leqslant p_1 + \cdots + p_{e-1} < j$. The latter statement therefore holds.

Let $A$ denote the $n \times m$ matrix defined by

$$a_{ij} = \begin{cases} \sqrt{(f * \mu)(y_j)} & \text{if } y_j \leqslant x_i, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

By Lemma 3.2 we have

$$\det(S)_f = \det(AA^{\mathrm{T}}). \tag{5.3}$$

Now let $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ denote the system of row vectors of $A$ and let $\{\beta_1,\beta_2,\ldots,\beta_n\}$ denote the orthogonalization system obtained from $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ by using the Gram–Schmidt orthogonalization process

$$\begin{cases} \beta_1 = \alpha_1, \\ \beta_k = \alpha_k - \sum_{i=1}^{k-1} \frac{\langle \alpha_k, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} \beta_i, \end{cases} \tag{5.4}$$

where $2 \leqslant k \leqslant n$. Let finally $B$ denote the $n \times m$ matrix having $\beta_i$'s as its rows. From the orthogonalization algorithm we find that there exists an invertible matrix $E$, which is the product of elementary matrices, such that $\det E = 1$ and $EA = B$. Thus

$$\det(AA^{\mathrm{T}}) = \det(E^{-1}BB^{\mathrm{T}}(E^{-1})^{\mathrm{T}}) = \det(BB^{\mathrm{T}}). \tag{5.5}$$

On the other hand, the set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is orthogonal. Thus

$$BB^{\mathrm{T}} = [\langle \beta_i, \beta_j \rangle] = \mathrm{diag}(\langle \beta_1, \beta_1 \rangle, \langle \beta_2, \beta_2 \rangle, \ldots, \langle \beta_n, \beta_n \rangle) \tag{5.6}$$

and

$$\det(BB^{\mathrm{T}}) = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle. \tag{5.7}$$

It follows from (5.3) and (5.5)–(5.7) that

$$\det(S)_f = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle. \tag{5.8}$$

From the definition of the matrix $A$ we see that

$$\begin{aligned} \alpha_1 &= \left( \sqrt{(f * \mu)(y_{1,1})}, \ldots, \sqrt{(f * \mu)(y_{1,p_1})}, \underbrace{0, \ldots, 0}_{p_2 + \cdots + p_n} \right), \\ \alpha_k &= \left( \underbrace{*, \ldots, *}_{p_1 + \cdots + p_{k-1}}, \sqrt{(f * \mu)(y_{k,1})}, \ldots, \sqrt{(f * \mu)(y_{k,p_k})}, \underbrace{0, \ldots, 0}_{p_{k+1} + \cdots + p_n} \right), \end{aligned} \tag{5.9}$$

where $2 \leqslant k \leqslant n$. By orthogonalizing we have

$$\beta_1 = \left( \sqrt{(f*\mu)(y_{1,1})}, \ldots, \sqrt{(f*\mu)(y_{1,p_1})}, \underbrace{0, \ldots, 0}_{p_2+\cdots+p_n} \right),$$

$$\beta_k = \left( \underbrace{*, \ldots, *}_{p_1+\cdots+p_{k-1}}, \sqrt{(f*\mu)(y_{k,1})}, \ldots, \sqrt{(f*\mu)(y_{k,p_k})}, \underbrace{0, \ldots, 0}_{p_{k+1}+\cdots+p_n} \right), \tag{5.10}$$

where $2 \leqslant k \leqslant n$. We find that the Gram–Schmidt process changes only numbers marked by the asterisk. Since $f \in C_S$, we have

$$\langle \beta_k, \beta_k \rangle \geqslant \sum_{t=1}^{p_k} (f*\mu)(y_{k,t}) = \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f*\mu)(z) > 0 \tag{5.11}$$

and

$$\det(S)_f = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle \geqslant \prod_{k=1}^{n} \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f*\mu)(z). \tag{5.12}$$

Therefore (5.1) holds.

Let $S$ be meet-closed. We prove by induction on $k$ that

$$\beta_k = \left( \underbrace{0, \ldots, 0}_{p_1+\cdots+p_{k-1}}, \sqrt{(f*\mu)(y_{k,1})}, \ldots, \sqrt{(f*\mu)(y_{k,p_k})}, \underbrace{0, \ldots, 0}_{p_{k+1}+\cdots+p_n} \right), \tag{5.13}$$

whenever $1 \leqslant k \leqslant n$. Since $\alpha_1 = \beta_1$, we have that (5.13) holds for $\beta_1$. Assume that (5.13) holds for $\beta_1, \beta_2, \ldots, \beta_{k-1}$, $1 < k \leqslant n$. Now consider $\beta_k$ and let $i$ be an index such that $p_1 + \cdots + p_{e-1} < i \leqslant p_1 + \cdots + p_e$ and let $1 \leqslant e \leqslant k-1$. By (5.10) we see that $\beta_1^{(i)} = \cdots = \beta_{e-1}^{(i)} = 0$. By the induction assumption we have $\beta_{e+1}^{(i)} = \cdots = \beta_{k-1}^{(i)} = 0$. Thus

$$\begin{aligned} \beta_k^{(i)} &= \alpha_k^{(i)} - \frac{\langle \alpha_k, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \beta_1^{(i)} - \cdots - \frac{\langle \alpha_k, \beta_{k-1} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle} \beta_{k-1}^{(i)} \\ &= \alpha_k^{(i)} - \frac{\langle \alpha_k, \beta_e \rangle}{\langle \beta_e, \beta_e \rangle} \beta_e^{(i)}. \end{aligned} \tag{5.14}$$

Since $x_i < x_j \Rightarrow i < j$ holds in $S$, we have either $x_e < x_k$ or $x_e \nleqslant x_k$. First assume that $x_e < x_k$. Now $y_{e,i} \leqslant x_e < x_k$; hence $\alpha_k^{(i)} = \beta_e^{(i)} = \sqrt{(f*\mu)(y_{e,i})}$. Since $i$ is chosen arbitrarily, we have $\langle \alpha_k, \beta_e \rangle = \langle \beta_e, \beta_e \rangle$. Now by (5.14) we have

$$\beta_k^{(i)} = \alpha_k^{(i)} - \frac{\langle \alpha_k, \beta_e \rangle}{\langle \beta_e, \beta_e \rangle} \beta_e^{(i)} = \alpha_k^{(i)} - \beta_e^{(i)} = 0.$$

We now assume that $x_e \nleqslant x_k$. Then $y_{e,1}, y_{e,2}, \ldots, y_{e,p_e} \nleqslant x_k$. Otherwise, there exists $s$, $1 \leqslant s \leqslant p_e$, such that $y_{e,s} \leqslant x_k$. Since $S$ is meet-closed, there exists $d < e$ such that $x_e \wedge x_k = x_d$. Since $y_{e,s} \leqslant x_e$, $y_{e,s} \leqslant x_k$ and $x_e \nleqslant x_k$, we have $y_{e,s} \leqslant x_d$, which contradicts $d < e$. So $y_{e,1}, y_{e,2}, \ldots, y_{e,p_e} \nleqslant x_k$ and $\alpha_k^{(i)} = 0$. Since $i$ is chosen arbitrarily, we have $\langle \alpha_k, \beta_e \rangle = 0$. Now by (5.14) we have

$$\beta_k^{(i)} = \alpha_k^{(i)} - \frac{\langle \alpha_k, \beta_e \rangle}{\langle \beta_e, \beta_e \rangle} \beta_e^{(i)} = 0.$$

Since $\beta_k^{(i)} = 0$ for $p_1 + \cdots + p_{e-1} < i \leqslant p_1 + \cdots + p_e$ and $1 \leqslant e \leqslant k-1$, we see by (5.14) that (5.13) holds for $\beta_k$. This completes the proof of (5.13). Now

$$\langle \beta_k, \beta_k \rangle = \sum_{t=1}^{p_k} (f * \mu)(y_{k,t}) = \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f * \mu)(z) > 0$$

and

$$\det(S)_f = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle = \prod_{k=1}^{n} \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f * \mu)(z).$$

Therefore, if $S$ is meet-closed, then the equality holds in (5.1).

Now let $S$ be a set such that the equality holds in (5.1). We show that $S$ is meet-closed. On the contrary, assume that $S$ is not meet-closed. Since $\{x_1\}$ is meet-closed, there exists minimal $a \geqslant 2$ such that $\{x_1, x_2, \ldots, x_{a-1}\}$ is meet-closed but $\{x_1, x_2, \ldots, x_a\}$ is not meet-closed. Now (5.13) holds for $\{x_1, x_2, \ldots, x_{a-1}\}$, that is,

$$\begin{aligned} \beta_1 &= \left( \sqrt{(f * \mu)(y_{1,1})}, \ldots, \sqrt{(f * \mu)(y_{1,p_1})}, \underbrace{0, \ldots, 0}_{p_2 + \cdots + p_n} \right), \\ &\;\vdots \\ \beta_{a-1} &= \left( \underbrace{0, \ldots, 0}_{p_1 + \cdots + p_{a-2}}, \sqrt{(f * \mu)(y_{a-1,1})}, \ldots, \sqrt{(f * \mu)(y_{a-1,p_{a-1}})}, \underbrace{0, \ldots, 0}_{p_a + \cdots + p_n} \right). \end{aligned} \tag{5.15}$$

Let $b$ be the minimal index such that $1 \leqslant b \leqslant a-1$ and $x_a \wedge x_b \notin S$. Clearly $b$ exists. Since $\downarrow S$ is lower-closed, it is meet-closed. Therefore $x_a \wedge x_b = y_{d,c}$, where $1 \leqslant d \leqslant b$ and $1 \leqslant c < p_d$ (if $c = p_d$, then $y_{d,c} = x_d$, which leads to a contradiction). We show that $d = b$. Otherwise, if $d < b$, we have $x_a \wedge x_d \in S$ by minimality of $b$. Let $x_a \wedge x_d = x_l$, $l \leqslant d$. Since $y_{d,c} \leqslant x_a$ and $y_{d,c} \leqslant x_d$, we have $y_{d,c} \leqslant x_l$.

Thus $d \leqslant l$. Since $l \leqslant d$, we have $l = d$. Thus $x_a \wedge x_d = x_d$ and $x_d \leqslant x_a$. Since $\{x_1, x_2, \dots, x_{a-1}\}$ is meet-closed but $\{x_1, x_2, \dots, x_a\}$ is not meet-closed, where $d < b < a$, we have $x_b \wedge x_d \in S$. In the same way let $x_b \wedge x_d = x_h$, $h \leqslant d$. Since $y_{d,c} \leqslant x_b$ and $y_{d,c} \leqslant x_d$, we have $y_{d,c} \leqslant x_h$. Thus $d \leqslant h$. Since $h \leqslant d$, we have $h = d$. Thus $x_b \wedge x_d = x_d$ and $x_d \leqslant x_b$. Now $x_d \leqslant x_a$ and $x_d \leqslant x_b$. We have therefore $x_d \leqslant x_a \wedge x_b = y_{d,c}$. This leads to a contradiction, because $y_{d,c} = x_d \in S$. Therefore $d = b$ and $x_a \wedge x_b = y_{b,c}$, where $1 \leqslant c < p_b$.

By (5.15) we have $\beta_1^{(i)} = \cdots = \beta_{b-1}^{(i)} = \beta_{b+1}^{(i)} = \cdots = \beta_{a-1}^{(i)} = 0$ and thus

$$\begin{aligned}\beta_a^{(i)} &= \alpha_a^{(i)} - \frac{\langle \alpha_a, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1^{(i)} - \cdots - \frac{\langle \alpha_a, \beta_{a-1} \rangle}{\langle \beta_{a-1}, \beta_{a-1} \rangle}\beta_{a-1}^{(i)} \\ &= \alpha_a^{(i)} - \frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle}\beta_b^{(i)} \end{aligned} \tag{5.16}$$

for all $p_1 + \cdots + p_{b-1} < i \leqslant p_1 + \cdots + p_b$. First assume that $\langle \alpha_a, \beta_b \rangle = 0$ and let $i = p_1 + \cdots + p_{b-1} + c$. Then by (5.16) we have $\beta_a^{(i)} = \alpha_a^{(i)} = \sqrt{(f * \mu)(y_{b,c})} > 0$ and

$$\langle \beta_a, \beta_a \rangle \geqslant (f * \mu)(y_{b,c}) + \sum_{i=1}^{p_a} (f * \mu)(y_{a,i}) > \sum_{\substack{z \leqslant x_a \\ z \nleqslant x_t \\ t < a}} (f * \mu)(z) > 0.$$

We now assume that $\langle \alpha_a, \beta_b \rangle \neq 0$ and let $i = p_1 + \cdots + p_b$. Then

$$\frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle} \neq 0.$$

Since $x_b \nleqslant x_a$, we have $\alpha_a^{(i)} = 0$. Then by (5.16) we have

$$\beta_a^{(i)} = -\frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle}\sqrt{(f * \mu)(x_b)} \neq 0$$

and

$$\begin{aligned}\langle \beta_a, \beta_a \rangle \geqslant{}& \left(\frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle}\right)^2 (f * \mu)(x_b) \\ &+ \sum_{i=1}^{p_a} (f * \mu)(y_{a,i}) > \sum_{\substack{z \leqslant x_a \\ z \nleqslant x_t \\ t < a}} (f * \mu)(z) > 0.\end{aligned}$$

In both cases we have

$$\det(S)_f = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle > \prod_{k=1}^{n} \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f * \mu)(z)$$

and this is a contradiction. Therefore $S$ is meet-closed. This completes the proof of Theorem 5.1. □

## 6. Upper bound for $\det(S)_f$

**Lemma 6.1.** *If $f \in C_S$, then $(S)_f$ is positive definite.*

**Proof.** Let $f \in C_S$. Then $(f * \mu)(z) > 0$ whenever $z \leqslant x_i$ and $x_i \in S$. Define $S_i = \{x_1, x_2, \ldots, x_i\}$, $1 \leqslant i \leqslant n$. Then by Theorem 5.1 we have

$$\det(S_i)_f \geqslant \prod_{k=1}^{i} \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f * \mu)(z) > 0,$$

where $1 \leqslant i \leqslant n$. Thus the principal minors of $(S)_f$ are positive. This completes the proof. □

**Lemma 6.2.** *Suppose that*

$$A = \begin{bmatrix} B & D \\ D^* & C \end{bmatrix}$$

*is a positive definite matrix that is partitioned so that B and C are square and non-empty, $D^*$ being the conjugate transpose of D. Then* $\det A \leqslant (\det B)(\det C)$.

Lemma 6.2 is known as Fisher's inequality and it can be found in [3]. Now we give an upper bound for $\det(S)_f$. Haukkanen provided the same result in [1].

**Theorem 6.1.** *If $f \in C_S$, then*

$$\det(S)_f \leqslant f(x_1) \cdots f(x_n). \tag{6.1}$$

We now provide a new upper bound for $\det(S)_f$. The new upper bound (6.2) is sharper than (6.1) if we choose $m = 2$. To see this we need Lemma 6.3, which is also needed in the proof of the new upper bound.

**Lemma 6.3.** *If $f \in C_S$, then $f(x_i) > 0$ for all $x_i \in S$. Furthermore, if $x < x_i$ and $x_i \in S$, then $f(x) < f(x_i)$.*

**Proof.** Let $f \in C_S$ and $x_i \in S$. Then $(f * \mu)(z) > 0$ for all $z \leqslant x_i$. Thus by Lemma 3.1 we have $f(x_i) = \sum_{z \leqslant x_i} (f * \mu)(z) > 0$. Let $x < x_i$. Then

$$\begin{aligned} f(x) &= \sum_{z \leqslant x} (f * \mu)(z) \\ &< (f * \mu)(x_i) + \sum_{z \leqslant x} (f * \mu)(z) \leqslant \sum_{z \leqslant x_i} (f * \mu)(z) = f(x_i). \end{aligned}$$

This completes the proof. □

**Theorem 6.2.** *If $f \in C_S$, then*

$$\det(S)_f \leqslant \frac{m!}{2} \left( 1 - \frac{f(x_{a_1} \wedge \cdots \wedge x_{a_m})^m}{f(x_{a_1}) \cdots f(x_{a_m})} \right) \prod_{k=1}^{n} f(x_k) \tag{6.2}$$

*whenever $1 \leqslant a_1 < \cdots < a_m \leqslant n$ and $2 \leqslant m \leqslant n$.*

**Proof.** Let $f \in C_S$. Define $U = \{x_{a_1}, x_{a_2}, \ldots, x_{a_m}\}$, where $1 \leqslant a_1 < \cdots < a_m \leqslant n$ and $2 \leqslant m \leqslant n$. Let $V = S \backslash U = \{x_{b_1}, x_{b_2}, \ldots, x_{b_{n-m}}\}$ with $x_{b_i} < x_{b_j} \Rightarrow i < j$. Define

$$A = \begin{bmatrix} (U)_f & D \\ D^{\mathrm{T}} & (V)_f \end{bmatrix},$$

where $D = [f(x_{a_i} \wedge x_{b_j})]$. By Lemma 6.1, $(S)_f$ is positive definite. Note that there is a permutation matrix $Q$ such that $A = Q^{\mathrm{T}}(S)_f Q$. Thus $\det(S)_f = \det A$ and $A$ is positive definite. Thus, by Lemma 6.2 and Theorem 6.1, we have

$$\det(S)_f \leqslant \left(\det(U)_f\right)\left(\det(V)_f\right) \leqslant \det(U)_f \prod_{k=1}^{n-m} f(x_{b_k}).$$

On the other hand,

$$\det(U)_f = \sum_{\pi=(p_1,p_2,\ldots,p_m)} (\operatorname{sgn}\pi) f(x_{a_1} \wedge x_{a_{p_1}}) \cdots f(x_{a_m} \wedge x_{a_{p_m}}),$$

where $\pi$ runs through the $m!$ permutations of $(1, 2, \ldots, m)$ and where $\operatorname{sgn}\pi = 1$ if $\pi$ is even and $\operatorname{sgn}\pi = -1$ if $\pi$ is odd (see [3, p. 8]). Obviously there are $m!/2$ permutations of each type. Let $\pi = (p_1, p_2, \ldots, p_m)$ be even. Since $x_{a_i} \wedge x_{a_{p_i}} \leqslant x_{a_i}$ and $f \in C_S$, by Lemma 6.3 we have $0 < f(x_{a_i} \wedge x_{a_{p_i}}) \leqslant f(x_{a_i})$ for all $1 \leqslant i \leqslant m$. Thus

$$f(x_{a_1} \wedge x_{a_{p_1}}) \cdots f(x_{a_m} \wedge x_{a_{p_m}}) \leqslant f(x_{a_1}) \cdots f(x_{a_m}).$$

Let $\pi = (p_1, p_2, \ldots, p_m)$ be odd. Since $x_{a_1} \wedge \cdots \wedge x_{a_m} \leqslant x_{a_i} \wedge x_{a_{p_i}}$ and $f \in C_S$, by Lemma 6.3 we have $0 < f(x_{a_1} \wedge \cdots \wedge x_{a_m}) \leqslant f(x_{a_i} \wedge x_{a_{p_i}})$ for all $1 \leqslant i \leqslant m$. Thus

$$\left[f(x_{a_1} \wedge \cdots \wedge x_{a_m})\right]^m \leqslant f(x_{a_1} \wedge x_{a_{p_1}}) \cdots f(x_{a_m} \wedge x_{a_{p_m}})$$

and

$$\begin{aligned}\det(U)_f &= \sum_{\pi=(p_1,p_2,\ldots,p_m)} (\operatorname{sgn}\pi) f(x_{a_1} \wedge x_{a_{p_1}}) \cdots f(x_{a_m} \wedge x_{a_{p_m}}) \\ &\leqslant \frac{m!}{2}\left(f(x_{a_1}) \cdots f(x_{a_m}) - \left[f(x_{a_1} \wedge \cdots \wedge x_{a_m})\right]^m\right).\end{aligned}$$

Therefore

$$\begin{aligned}\det(S)_f &\leqslant \left(\det(U)_f\right)\left(\det(V)_f\right) \\ &\leqslant \frac{m!}{2}\left(f(x_{a_1}) \cdots f(x_{a_m}) - \left[f(x_{a_1} \wedge \cdots \wedge x_{a_m})\right]^m\right) \prod_{k=1}^{n-m} f(x_{b_k}) \\ &= \frac{m!}{2}\left(1 - \frac{f(x_{a_1} \wedge \cdots \wedge x_{a_m})^m}{f(x_{a_1}) \cdots f(x_{a_m})}\right) \prod_{k=1}^{n} f(x_k).\end{aligned}$$

This completes the proof. □

## 7. Inverse of $(S)_f$

**Theorem 7.1.** *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a meet-closed set and let $f \in C_S$. Then $(S)_f$ is invertible and*

$$((S)_f^{-1})_{ij} = \sum_{\substack{x_i \leqslant x_k \\ x_j \leqslant x_k}} \frac{1}{\sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f * \mu)(z)} \mu_S(x_i, x_k)\mu_S(x_j, x_k), \tag{7.1}$$

*where $\mu_S = \zeta_S^{-1}$ and $\zeta_S$ is the restriction of $\zeta$ on $S \times S$.*

**Proof.** Define the order ideal as $\downarrow S = \{y_1, y_2, \ldots, y_m\}$ and define the matrix $A$ as in (5.2). Then $(S)_f = AA^{\mathrm{T}}$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ denote the system of row vectors of $A$ and let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ denote the orthogonalization system obtained from $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ by using the Gram–Schmidt orthogonalization prosess. Let $B$ denote the $n \times m$ matrix having $\beta_i$'s as its rows. Since $S$ is meet-closed, we have

$$BB^{\mathrm{T}} = \operatorname{diag}\left(\sum_{\substack{z \leqslant x_1 \\ z \nleqslant x_t \\ t<1}} (f * \mu)(z), \ldots, \sum_{\substack{z \leqslant x_n \\ z \nleqslant x_t \\ t<n}} (f * \mu)(z)\right).$$

By (2.1) we see that $BB^{\mathrm{T}}$ is invertible and

$$(BB^{\mathrm{T}})^{-1} = \mathrm{diag}\left( \frac{1}{\sum_{\substack{z \leqslant x_1 \\ z \nleqslant x_t \\ t<1}} (f * \mu)(z)}, \ldots, \frac{1}{\sum_{\substack{z \leqslant x_n \\ z \nleqslant x_t \\ t<n}} (f * \mu)(z)} \right).$$

By algorithm (5.4) we have

$$\alpha_1 = \beta_1,$$
$$\alpha_k = \beta_k + \sum_{i=1}^{k-1} \frac{\langle \alpha_k, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} \beta_i,$$

where $2 \leqslant k \leqslant n$. We note that $EB = A$, where

$$E = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} & 1 & 0 & \cdots & 0 \\ \frac{\langle \alpha_3, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} & \frac{\langle \alpha_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\langle \alpha_n, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} & \frac{\langle \alpha_n, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} & \frac{\langle \alpha_n, \beta_3 \rangle}{\langle \beta_3, \beta_3 \rangle} & \cdots & 1 \end{bmatrix}.$$

Since $S$ is meet-closed, we know by the proof of Theorem 5.1 (see the discussion after (5.14)) that $\langle \alpha_i, \beta_j \rangle = \langle \beta_j, \beta_j \rangle$ if $x_j < x_i$, and $\langle \alpha_i, \beta_j \rangle = 0$ if $x_j \nleqslant x_i$. Thus $E = [\zeta_S(x_i, x_j)]^{\mathrm{T}}$ and so $E^{-1} = [\mu_S(x_i, x_j)]^{\mathrm{T}}$. Then $(S)_f = AA^{\mathrm{T}} = EBB^{\mathrm{T}}E^{\mathrm{T}}$ and $(S)_f^{-1} = (E^{-1})^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}E^{-1}$. Therefore

$$((S)_f^{-1})_{ij} = \sum_{\substack{x_i \leqslant x_k \\ x_j \leqslant x_k}} \frac{1}{\sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f * \mu)(z)} \mu_S(x_i, x_k)\mu_S(x_j, x_k).$$

This completes the proof. □

Haukkanen [1] has proved a similar formula for $(S)_f^{-1}$ when $S$ is lower-closed. He also mentions the possibility of proving this more general result by using $\mu_S$.

## References

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